Normal forms of Poisson structures near a symplectic leaf

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Abstract

In this paper, we show how one can handle the formalism developped by Yurii Vorobjev in order to give general results about the problems of linearisation and of normal form of a Poisson structure in the neighborhood of one of its symplectic leaves.

1 Coupling tensors

Here, we summarize the tools constructed by Vorobjev [Vo] that will be useful for our purpose. In order to study a Poisson structure Π on the neighborhood of one of its compact leaves S, we can choose a tubular neighborhood $p: E \longmapsto S$, seen as a vector bundle, and consider that Π is defined on E. Denote n the rank of E, and $2s$ the dimension of S, while $\mathcal{X}(S)$ and $\Omega(S)$ denote respectively the $C^\infty(S)-$modules of vector fields and of $1-$forms on S. Let $Ver := \ker p_\star \subset TE$ denote the sub-bundle of vertical vector fields, and $Ver^0 \subset T^\star E$ its annihilator.

As Π is non-degenerate on S, one can always choose E such that that it is *horizontally non-degenerate*, that is, verifies:

$$\left\{ \begin{array}{ccc} \Pi^\sharp(Ver^0) \cap Ver & = & \{0\} \\ \dim\, \Pi^\sharp(Ver^0) & = & \dim\, S \\ Im\Pi^\sharp_{|S} & = & TS, \end{array} \right. \tag{1.1}$$

where, as usual, $\Pi^\sharp : T^\star E \longmapsto TE$ denotes the natural contraction $\alpha \longmapsto i_\alpha \Pi$. By putting $Hor := \Pi^\sharp(Ver)$ one defines a subbundle of TE complementary to Ver, or equivalently, an *Ehresmann connection*, that is $\Gamma \in \Omega(E) \bigotimes_{C^\infty(S)} Ver$ such that $\Gamma(X_V) = X_V$ for all $X_V \in Ver$, Γ being just the projection onto Ver with kernel Hor. For any Ehresmann connection, given $X \in \mathcal{X}(S)$, there exists a unique section of Hor that $p-$projects onto

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X, one calls it the *horizontal lift* of X and denotes it $hor(X)$. The curvature of Γ is an element of $\Omega^2(S) \otimes_{C^\infty(S)} Ver$, defined as $Curv_\Gamma(u,v) := [horu, horv] - hor[u,v]$. Also, the covariant derivative attached to Γ, $\partial_\Gamma : \Omega^k(S) \bigotimes_{C^\infty(S)} C^\infty(E) \longmapsto \Omega^{k+1}(S) \bigotimes_{C^\infty(S)} C^\infty(E)$ defined by the following:

$$\begin{aligned}(\partial_\Gamma \mathcal{F})(u_0, u_1, \ldots, u_k) := &\textstyle\sum_{i=0}^k (-1)^i \mathcal{L}_{hor(u_i)} \mathcal{F}(u_0, u_1, \ldots, \hat{u}_i, \ldots, u_k) \\ &+ \textstyle\sum_{0 \le i < j \le k} (-1)^{i+j} \mathcal{F}([u_i, u_j], u_0, u_1, \ldots, \hat{u}_i, \ldots, \hat{u}_j, \ldots, u_k).\end{aligned}$$

Geometric data on E are defined to be triples $(\Gamma, \mathcal{V}, \mathcal{F})$ with Γ an Ehresmann connection, $\mathcal{V} \in \Lambda^2 Ver$ a vertical 2-vector field on E, and $\mathcal{F} \in \Omega^2(S) \bigotimes_{C^\infty(S)} C^\infty(E)$ a *non-degenerate* 2-form on S with values in $C^\infty(E)$. The first result of Vorobjev consists into describing any horizontally non-degenerate $2-$vector field on E as a geometric data.

Proposition 1.1. *Geometric data on E are equivalent to horizontally non-degenerate 2-vector fields*

proof: Given a non-degenerate $2-$vector field Π on E (not supposed to be Poisson), we define Γ to be the Ehresmann connection with kernel $Hor := \Pi^\sharp(Ver)$, as explained above. Let $\Gamma^{\Lambda 2} : \Lambda^2 TE \longmapsto Ver$ the external second power of Γ, define $\mathcal{V} \in \Lambda^2 Ver$ as $\mathcal{V} := \Gamma^2(\Pi)$, there only remains to define $\mathcal{F}$.

As Π is horizontally non-degenerate, $\Phi := \Pi^\sharp_{|Ver^0} \longmapsto Hor$ is an isomorphism, so that one can define $\mathcal{F} \in \Omega^2(S) \bigotimes_{C^\infty(S)} C^\infty(E)$ by requiring:

$$\mathcal{F}(u,v) = - < \Phi^{-1}(horu), horv >, \quad \forall u, v \in \mathcal{X}(S).$$

We refer the interested reader to [Vo] for the construction of the non-degenerate vector field given a geometric data $(\Gamma, \mathcal{V}, \mathcal{F})$, prefering to describe the situation locally. □

Let $(x_1 \ldots x_{2s})$ be a coordinate chart on an open neighborhood U of S such that $p^{-1}(U) = U \times \mathbb{R}^n$, with $(y_1 \ldots y_n)$ linear coordinates on the fibers. We can locally express Π as:

$$\Pi = \sum_{i,j=1\ldots 2s} \Pi^X_{i,j} \partial x_i \wedge \partial x_j + \sum_{i=1\ldots 2s, k=1\ldots n} 2\Pi^{XY}_{i,k} \partial x_i \wedge \partial y_k + \sum_{k,l=1\ldots n} \Pi^Y_{k,l} \partial y_k \wedge \partial y_l,$$

with $\Pi^X_{i,j}$, $\Pi^{XY}_{i,k}$, $\Pi^Y_{k,l} \in C^\infty(p^{-1}(U))$ satisfying $\Pi^X_{i,j} = -\Pi^X_{j,i}$, and $\Pi^Y_{k,l} = -\Pi^Y_{l,k}$. The horizontal non-degeneracy condition means that the matrix $(\Pi^X_{i,j})$ is invertible, let $({\Pi^X_{i,j}}^{-1})$ denote its inverse, we have:

$$\begin{aligned} Hor &= Span \left\{ \partial x_i + \textstyle\sum_{j=1\ldots 2s, k=1\ldots n} {\Pi^X_{i,j}}^{-1} \Pi^{XY}_{j,k} \partial y_k, \ i = 1 \ldots n \right\} \\ &= Span \left\{ \partial x_i + \textstyle\sum_{k=1\ldots n} \beta_{i,k} \partial y_k, \ i = 1 \ldots n \right\}, \end{aligned}$$

where $\beta_{i,k} := \sum_{j=1\dots 2s} \Pi^{X}_{i,j}{}^{-1}\Pi^{XY}_{j,k}$, so that we get the following local formula for horizontal lifts:

$$hor(\partial x_i) = \partial x_i + \sum_{k=1\dots n} \beta_{i,k}\partial y_k.$$

Let us denote horizontal lifts $X_i := hor(\partial x_i)$ for $i = 1 \dots 2s$. Putting

$$\mathcal{V}_{p,q} := \Pi^{Y}_{p,q} - \frac{1}{2}\sum_{l=1\dots 2s} \beta_{l,p}\Pi^{XY}_{l,q} - \beta_{l,q}\Pi^{XY}_{l,p},$$

allows us to include mixed terms $\Pi^{XY}_{i,k}\partial x_i \wedge \partial y_k$ into basic ones $\Pi^{X}_{i,j}\partial x_i \wedge \partial x_j$:

$$\Pi = \sum_{i,j=1\dots 2s} \Pi^{X}_{i,j} X_i \wedge X_j + \sum_{k,l=1\dots n} \mathcal{V}_{k,l}\partial y_k \wedge \partial y_l.$$

This way, $\mathcal{V}_{p,q}$ appear to be the local coefficients for $\mathcal{V}$, while $\mathcal{F}$ expresses as:

$$\mathcal{F}(\partial x_i, \partial x_j) = -\frac{1}{2}\Pi^{X}_{i,j}{}^{-1}.$$

Note that the above decomposition

$$\Pi = \Pi_H + \mathcal{V} \tag{1.2}$$

with $\Pi_H = \sum_{i,j=1\dots 2s} \Pi^{X}_{i,j} X_i \wedge X_j \in \Lambda^2 Hor$ is *global* on E.

Next proposition gives the Poisson condition in terms of geometric data.

Proposition 1.2. *Let Π be a horizontally non-degenerate 2-vector field on E with corresponding geometric data $(\Gamma, \mathcal{V}, \mathcal{F})$. Then Π is Poisson if and only if*

$$\begin{array}{rcll}
[\mathcal{V}, \mathcal{V}] & = & 0, & & (i) \\
\mathcal{L}_{hor(X)}\mathcal{V} & = & 0 & \forall X \in \mathcal{X}(S), & (ii) \\
\partial_\Gamma \mathcal{F} & = & 0, & & (ii) \\
Curv_\Gamma(u,v) & = & \mathcal{V}^\sharp(d\mathcal{F}(u,v)) & \forall u, v \in \mathcal{X}(S). & (iv)
\end{array}$$

The proof can be carried out with a long calculus, so I refer mistrustful (and courageous) readers to my Phd. Thesis [Br].

Proposition 1.3. (Semi-local splitting) *Let Π be an horizontally non-degenerate Poisson, with corresponding geometric data $(\Gamma, \mathcal{V}, \mathcal{F})$. In the decomposition 1.2:*

$$\Pi = \Pi_H + \mathcal{V},$$

the 2-vector field Π_H is Poisson if and only if Γ has null curvature.

The proof lies in [Br], and is left to the reader.

Remark 1.4. The problem of finding a tubular neighborhood $p : E \longmapsto S$ such that Π_H is Poisson is the semi-local analog of Weinstein's splitting theorem (see [We1],[We2]), but may not hold if, for exemple, E does not admit connections without curvature. However, some positive answer will be given later on.

Before giving conditions under which we will be able to construct some homotopy between two horizontally non-degenerate Poisson structures having same vertical part, let us precise some notations.

Given $\phi \in \Omega^1(S) \otimes_{C^\infty(S)} C^\infty(E)$, one defines $\mathcal{V}^\sharp(d\phi)_h \in \Omega^1(E) \otimes_{C^\infty(E)} \chi(Ver)$ by putting

$$\mathcal{V}^\sharp(d\phi)_h(X) := \mathcal{V}^\sharp_x(d_x\phi(p_\star X)) \in Ver_x \quad \forall x \in E, \quad \forall X \in T_xE.$$

It is easily verified that this definition point by point assures $C^\infty(E)$-linearity.

Besides, given $\phi_1, \phi_2 \in \Omega^1(S) \otimes_{C^\infty(S)} C^\infty(E)$ two 1-forms on S with values in $C^\infty(E)$, it is defined $\{\phi_1, \phi_2\}_\mathcal{V}$ to be the 2-form on S with values in $C^\infty(E)$ (that is $\{\phi_1, \phi_2\}_\mathcal{V} \in \Omega^2(S) \otimes_{C^\infty(S)} C^\infty(E)$) such that for all $u_1, u_2 \in \chi(S)$:

$$\{\phi_1, \phi_2\}_\mathcal{V}(u_1, u_2) := \mathcal{V}\big(d\phi_1(u_1), d\phi_2(u_2)\big) - \mathcal{V}\big(d\phi_1(u_2), d\phi_2(u_1)\big).$$

Proposition 1.5. *Let Π and Π' two horizontally non-degenerate Poisson structure on some vector bundle E over a compact base S, with corresponding geometric datas $(\Gamma, \mathcal{V}, \mathcal{F})$ et $(\Gamma', \mathcal{V}', \mathcal{F}')$. Suppose that Π and Π' have same vertical part and coincide on S:*

$$\begin{aligned} \mathcal{V}' &= \mathcal{V} \\ \mathcal{F}'(u_1, u_2)_{|S} &= \mathcal{F}(u_1, u_2)_{|S} \quad \forall u_1, u_2 \in \mathcal{X}(S). \end{aligned}$$

If there exists some 1-form with values in $C^\infty(E)$, $\phi \in \Omega^1(S) \otimes_{C^\infty(S)} C^\infty(E)$ such that:

$$\begin{aligned} \Gamma' &= \Gamma - \mathcal{V}^\sharp(d\phi)_h \\ \mathcal{F}' &= \mathcal{F} + \partial_\Gamma \phi + \tfrac{1}{2}\{\phi, \phi\}_\mathcal{V}, \end{aligned}$$

then there exists neighborhoods $\mathcal{U}, \mathcal{U}'$ of S in E and a diffeomorphism $\Phi : \mathcal{U} \longmapsto \mathcal{U}'$ such that

$$\begin{aligned} \Phi_\star \Pi &= \Pi' \\ \Phi_{|S} &= Id_S. \end{aligned}$$

Let' s just sketch the proof (see [Vo] for the details). It consists in constructing some homotopy between Π and Π', that is just the flow of a time-dependant vector field X_t linked to ϕ via the non-degeneracy condition for $\mathcal{F}$. The conditions on ϕ are exactly the ones for X_t to assure that it will tract Π to Π'.

2 Linearisation near a symplectic leaf

The problem of linearising a Poisson structure that vanishes at one point was studied in [Co], [Du1], [Du2], [Du3], [D-Zu]... In this section, we want to see how these results extend to a full neighborhood of a fixed symplectic leaf, in the most general situation possible.

Proposition 2.1. (Linearisation of the vertical part) *Let Π an horizontally non-degenerate Poisson structure on some vector bundle $p : E \longmapsto S$ over a compact base S, with corresponding geometric data $(\Gamma, \mathcal{V}, \mathcal{F})$. Suppose that, fore some $x \in S$, the germ of $\mathcal{V}$ at x is linearisable (as a Poisson structure on E_x), then the germ of Π along S is equivalent to the one of a Poisson structure with associated geometric data $(\Gamma', \mathcal{V}^{(1)}, \mathcal{F}')$, for some Γ' and $\mathcal{F}'$. Here $\mathcal{V}^{(1)}$ denotes the linear part of $\mathcal{V}$.*

proof: The vertical part is locally trivial (as $\mathcal{V}$ is invariant by a connection according to condition (ii)) so the proposition is trivial if S is some ball. For more general S, the problem is to paste local linearising charts. The argument is the following: instead of linearising directly $\mathcal{V}$, one constructs a vector field $Z \in \mathcal{X}(E)$ such that:

$$[Z, \mathcal{V}] = -\mathcal{V} \tag{2.1}$$

$$Z^{(1)} = L, \tag{2.2}$$

where $Z^{(1)}$ denotes the part of order one in the fibers of Z , and $L = \sum_{i=1\ldots n} y_i \partial y_i$ the so called *Liouville vector field* (note that the form $\sum_{i=1\ldots n} y_i \partial y_i$ does not depend on the chosen trivialisation of the vector bundle E as it is equivalent to requiring that the flow ϕ_t of $Z^{(1)}$ is multiplication by $\exp(t)$). The vector field Z turns out to be *globally* linearisable, so that, up to a diffeomorphism defined on a some neighborhood of S, we have

$$[L, \mathcal{V}] = -\mathcal{V}.$$

It is then a basic fact that this last condition forces $\mathcal{V}$ itself to be fiberwise linear.

There remains to construct Z: let $(U_i)_{i \in I}$ some finite cover of S with U_i open balls of S, that admit trivialisations $\psi_i : p^{-1}(U_i) \longmapsto \mathcal{U}_i \times \mathbb{R}^n$ of the vector bundle E, and $(\rho_i)_{i \in I}$ some subordinate partition of unity. The local triviality of $\mathcal{V}$, both with the linearisability condition enables to construct vector fields Z_i over $p^{-1}(U_i)$ such that

$$\begin{aligned} [Z_i, \mathcal{V}] &= -\mathcal{V} \\ Z_i^{(1)} &= L. \end{aligned}$$

Let $Z := \sum_{i \in I} \rho_i Z_i$, then 2.2 is easily verified, and 2.1 holds because ρ_i are basic and $\mathcal{V}$ vertical. $\square$

Proposition 2.2. (Changing the connection) *Let Π an horizontally non-degenerate Poisson structure on some vector bundle $p : E \longmapsto S$ over a compact base S, with corresponding geometric data $(\Gamma, \mathcal{V}, \mathcal{F})$. Suppose that the first Poisson cohomology space $H^1(\mathcal{V}_x)$ of $\mathcal{V}$ at some $x \in S$ is trivial, and let Γ' some connection leaving $\mathcal{V}$ invariant.*

Then the germ of Π along S is equivalent to the one of a Poisson structure with associated geometric data $(\Gamma', \mathcal{V}, \mathcal{F}')$, for some $\mathcal{F}'$.

proof: According to 1.5, we only have to construct some $\phi \in \Omega^1(S) \otimes_{C^\infty(S)} C^\infty(E)$ such that:

$$\Gamma = \Gamma' + \mathcal{V}^\sharp(d\phi)^h.$$

Keeping the same notations as before, over $p^{-1}(U_j)$ Γ and Γ' take the form

$$\begin{array}{rcl}\Gamma(\partial x_i) & = & -\sum_{k=1}^n \beta_{i,k}\partial y_k \\ \Gamma'(\partial x_i) & = & -\sum_{k=1}^n \beta'_{i,k}\partial y_k.\end{array}$$

As Γ and Γ' are supposed to leave $\mathcal{V}$ invariant, we have

$$[\sum_{k=1}^n (\beta_{i,k} - \beta'_{i,k})\partial y_k, \mathcal{V}] = 0, \quad \forall i = 1 \dots 2s.$$

which means that $\sum_{k=1}^n (\beta_{i,k} - \beta'_{i,k})\partial y_k$ is a 1-cocycle for the Poisson cohomology of $\mathcal{V}$ (with parameters in U_i). By hypothesis, there exist $\phi_i^j \in C^\infty(p^{-1}(U_j))$ such that for all $i = 1 \dots 2s$,

$$[\mathcal{V}, \phi_i^j] = \sum_{k=1}^n (\beta_{i,k} - \beta'_{i,k})\partial y_k.$$

Let $\phi^j := \sum_{i=1 \dots 2s} \phi_i \otimes dx_i$, one gets on $p^{-1}(U_j))$ some $1-$form such that

$$\Gamma = \Gamma^{(1)} + \mathcal{V}^\sharp(d\phi^j)^h.$$

It is easily verified that, as ρ_i are basic, $\phi := \sum_{j \in J} \rho_j \phi^j$ gives the desired 1-form.$\square$

We are now handing enough tools to state the following theorem.

Theorem 2.3. (Semi-local linearisation of Poisson structures) *Let Π an horizontally non-degenerate Poisson structure on some vector bundle $p : E \longmapsto S$ over a compact base S, with corresponding geometric data $(\Gamma, \mathcal{V}, \mathcal{F})$. Suppose that the germ of $\mathcal{V}$ at x is linearisable (as a Poisson structure on E_x) for some $x \in S$ and that the first Poisson cohomology space $H^1(\mathcal{V}_x^{(1)})$ of $\mathcal{V}^{(1)}$ is trivial.*

Then the germ of Π along S is equivalent to the one of a Poisson structure with associated geometric data $(\Gamma^{(1)}, \mathcal{V}^{(1)}, \mathcal{F})$. Here $\mathcal{V}^{(1)}$ and $\Gamma^{(1)}$ respectively denote the linear parts of $\mathcal{V}$ and Γ.

proof: First apply proposition 2.1: up to a diffeomorphism, Π has associated geometric data $(\Gamma, \mathcal{V}^{(1)}, \mathcal{F})$, then Γ locally takes the form

$$hor(\partial x_i) = \partial x_i + \sum_{j=1\ldots n} \big(\sum_{k=1\ldots n} \beta_{i,j}^k(x) y_k + \hat{\beta}_{i,j}(x,y) \big) \partial y_j,$$

with $\hat{\beta}_{i,j}$ of order higher than one in the y_i-variables. Just isolating terms of order one in the equation

$$\mathcal{L}_{hor(\partial x_i)} \mathcal{V}^{(1)} = 0,$$

one sees that $\Gamma^{(1)}$ leaves $\mathcal{V}$ invariant, so we only have to apply proposition 2.2.$\square$

We can immediately derive the following corollaries.

Corollary 2.4. (Semi-local non-degeneracy of Poisson structures) *Let* Π *an horizontally non-degenerate Poisson structure on some vector bundle* $p : E \longmapsto S$ *over a compact base* S*, with corresponding geometric data* $(\Gamma, \mathcal{V}, \mathcal{F})$*. Suppose that the vertical linear part* $\mathcal{V}_x^{(1)}$ *at some* $x \in S$ *is non-degenerate and that the first Poisson cohomology space* $H^1(\mathcal{V}_x^{(1)})$ *of* $\mathcal{V}^{(1)}$ *is trivial.*

Then the germ of Π *along* S *is equivalent to the one of a Poisson structure with associated geometric data* $(\Gamma^{(1)}, \mathcal{V}^{(1)}, \mathcal{F}')$ *for some* $\mathcal{F}'$*.*

Corollary 2.5. (Semi-local non-degeneracy of transversally compact semi-simple Poisson structures) *Let* Π *an horizontally non-degenerate Poisson structure on some vector bundle* $p : E \longmapsto S$ *over a compact base* S*, with corresponding geometric data* $(\Gamma, \mathcal{V}, \mathcal{F})$*. Suppose that the vertical linear part* $\mathcal{V}_x^{(1)}$ *at some* $x \in S$ *is associated to a compact semi-simple Lie algebra.*

Then the germ of Π *along* S *is equivalent to the one of a Poisson structure with associated geometric data* $(\Gamma^{(1)}, \mathcal{V}^{(1)}, \mathcal{F}')$ *for some* $\mathcal{F}'$*.*

3 Semi-local splitting

Last theorem enables us to write certain Poisson structures Π as

$$\Pi = \Pi_H + \mathcal{V}^{(1)}$$

in a neighborhood of some leaf S, with Π_H inducing some linear conection on E. That' s what we mean by *global linearisation*. But up to now, we didn' t care about the 2-form $\mathcal{F}$, the first question being: what can we expect of it ? Some motivation is given by proposition 1.3, as we can see in proposition 1.2 that the curvature of Γ is intimely linked to $\mathcal{F}$. We already pointed out the necessary condition that E shall admit some connection without curvature, so this condition is not restrictive in the following proposition.

Theorem 3.1. *Let Π an horizontally non-degenerate Poisson structure on some vector bundle $p : E \longmapsto S$ over a compact base S, with corresponding geometric data $(\Gamma, \mathcal{V}, \mathcal{F})$. Suppose that the vertical part $\mathcal{V}_x$ at some $x \in S$ has trivial first Poisson cohomology space $H^1(\mathcal{V}_x)$, and that there exists some Ehresmann connection Γ' without curvature leaving $\mathcal{V}$ invariant.*

Then, up to a diffeomorphism on a neighborhood of S, Π decomposes into

$$\Pi = \Pi_H + \mathcal{V},$$

with Π_H and $\mathcal{V}$ Poisson structures such that Π_H is regular and $\mathcal{V}$ vertical, vanishing on S.

proof: Just apply propositions 2.2 and 1.3.

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